Asian Resonance

An Algorithamic Approach to Permutation Graphs and Its Complexities

Abstract

Permutation graph was first introduced by Chartrand and Harary in 1967 and their purpose was to study the cycle permutation graph. This paper definespermutation graph and study its properties and characterisationthrough theorems .It alsodiscuss characterisation of permutation labelling by a theorem .It gives real life application of permutation graph as a class of intersection graphs. It gives a sorting permutation using queues in parallel. It also gives canonical colouring of a permutation graph which gives a minimal colouring.This has given through an algorithm and discuss its complexity.

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Introduction

Let π be permutation of the numbers 1,2,3,....n .We can think of π as a sequence $[\pi_1\ \pi_2 \ldots\ldots\ldots \pi_n]$, so for example, the permutation π = [4,3,6,1,5,2] has $\pi_1 = 4$ $\pi_2 = 3$ etc. Notice that $(\pi^{-1})_i$, denoted here as π_i^{-1} is the position in the sequence where the number i can be found. For our example $\pi^{-1}{}_4 = 1$, $\pi_3^{-1} = 2$ etc.We can construct an undirected graph G[π] from π in the following manner ; G[π] have vertices numbered from 1. to n , two vertices are joined by an edge if the larger of their corresponding numbers is to the left of the smaller in π. (That is, they occur of their proper order reading left to right)

Basic definitions and notations

i) Permutation graphs: Let π be a permutation of the numbers 1,2,……….n. The graph G[π] = (V,E) is defined with V = (1,2,….. n)and i j ∈ E = (i-j) $(\pi^{-1}{}_i - \pi^{-1}{}_j) < 0$.
An undirected graph G is called a permutation graph if there exists a permutation π such that G ≅ G[π].
ii) Transitive Orientation Property: Each edge can assigned a one way direction in such a way that the resulting oriented graph (V,F) satisfies the following conditions: (ab)∈ F and (bc)∈ F imply (ac) ∈ F ∀a,b,c∈ F.
iii) Comparability graph : An undirected graph which is transitively orient able is called a Comparability graph.
iv) ChoromaticNumber :Choromatic number is the smallestpossible proper colouring of a graph .

Characterizing Permutation Graphs.[12]

Permutation graphs have many interesting properties when we reverse the sequence π, each pair of numbers which occurred in the correct order in π is now in the wrong order, and vice versa. Thus, the permutation graph we obtain is the complement of G[π] , If π^p is the permutation by reversing the sequence π ,then $G[\pi^p] = \overline{G[\pi]}$. This shows that the complement of a permutation graph is also a permutation graph. Another property of the graph G[π] is that it is transitively orient able. Iforient each edge towards its larger endpoints, than we will obtain a transitive orientation F. For , suppose (I,j) ∈F anf (j,k) ∈ F then i < j< k, and $\pi_i^{-1} > \pi_j^{-1} > \pi_k^{-1}$, which implies that (ik) ∈ F.

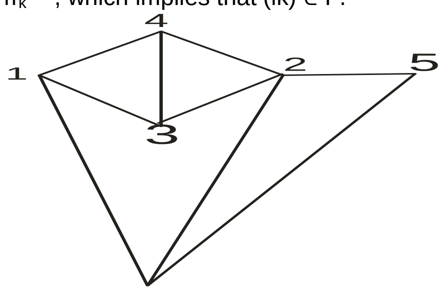


Figure 1. The graph G [4,3,6,l,5,2]

Theorem 1 [3]

An undirected graph G is a permutation graph ifand only if G and$\bar{G}$ are comparability graphs

Proof:

Let G be a permutation graph. Then by definition G = G[π] . Then G is a comparability graph since G[π] has a transitive orientation. In the same way $\bar{G}$is a comparability graph since$\bar{G}$ = G[π^p] . Conversely, let (V,F_1) and (V,F_2) he transitive orientation of G = (V,E) and $\bar{G}$ = (V, $\bar{E}$} , respectively, We claim that (V, $F_1 + F_2$) is an acyclic orientation of the complete graph (V,E +$\bar{E}$). For suppose $F_l + F_2$ had a cycle [v_o,v_1,…v_l,_,v_o} of the minimum length ℓ. ifℓ> 3, than the cycle can be shortened either by (v_o v_2) or (v_2v_o), contradicting minimality of ℓ. . If ℓ= 3, then at least this of the edges of the cycle are in the same F_1, implying that F_1 is not transitive Thus (V,F_1 + F_2) is acyclic. Similarly we can prove that (V,$F_1^{-1} + F_2$) is acyclic.

We complete the proof by constructing a permutation that π such that G = G[π]. An acyclic orientation of a complete graph is transitive, and it determines a unique linear ordering of the vertices, Consider the following steps:

Step 1 = Label the vertices according to the order determined by $F_l + F_2$. That is, the vertex x of in-degree i-1. gets label L(x) = i

Step 2 = Label the vertices according to the order determined by F_1^{-1} + F_2 That is the vertex x of in degree i – 1 gets the label L(x) = i

Notice that(xy) ∈ E ⇔ {L(x) – L(y)} { L$'$ (x) – L$'$(y)} < 0 …….(1) since it is the edges of E which have their orientationreversed between step 1 and step 2.

Step, 3 : Define π as follows; For each vertex x, if L(x) = i, than if π_i^{-1} = L'(x). This relationship is depicted in Figure 2. Therefore, by (1), π is the desired permutation and L is the desiredisomorphism.

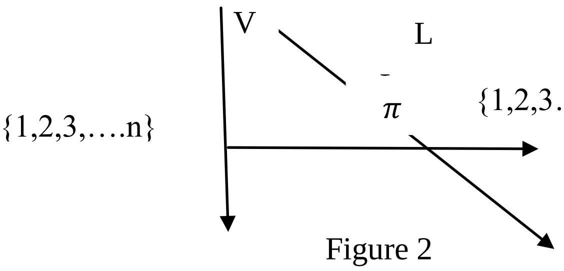
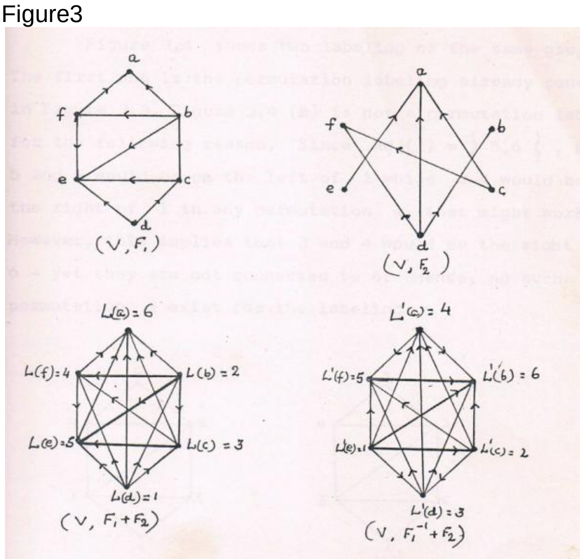


Figure 2

The above theorem suggests an algorithm for recognizing permutation graphs, namely, applying the transitive orientation algorithm to the graph and to its complement. If we can find a transitive orientation then the graph is a permutation graph. To find a suitable permutation we can follow the procedure in the proof of the theorem. Figure3. shows the construction of permutation π = [5,3,1,6,4,2] fromthe transitive orientation F_1 and F_2. The entire method requires 0(n^3) time and O(n^2) space for a graph with n vertices.[8],[9].

Figure3



Permutation Labelling

Labellingour problem is testing whether a given labelling of the vertices of graph is a permutation labelling. Let G = (Y.E) be an undirected graph, and let L : Y→ {l,,2,3 ,... n} be a bijection labelling the vertices., We call L a permutation labelling if there exists a permutation of {l,2……….n} such as xy∈ E ⇔ { L(x) - L(y) } {π^{-1} (L(x) - π^{-1} (L(y) } < 0 clearly, G is a permutation graph if and only if it has at least one permutation labelling.Figure 4 shows two labelling of the same graph.

The first one is the permutation labelling already constructed in Figure 3. Figure 4 (b) is not a permutation labelling for the following reason, Since Adj{1) = {5,6}both 5 and 6 would be on the left of 1while 2-4 would be on the right of 1 in any permutation π that might work. However, this implies that 3 and 4 would be the right of 6 - yet they are not connected to 6. Hence, no such permutation π exists for the labelling

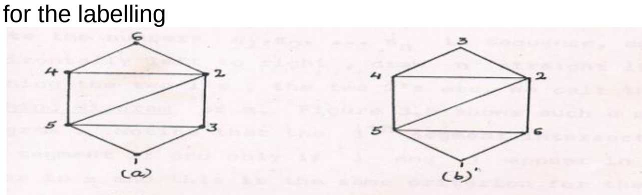


Figure 4

We give the following theorem without proof for the characterization of permutation labelling.

Theorem 2 [5]

Let G = (V, E) be an undirected graph. A bijectionL : V → {1,2,……..n) is a permutation labelling of G if and only if the mapping F : x→ L(x) - D^{--} (x) + D^+ (x) (x ∈ V) is an injection, where

D^{--}(x) = |{ Y∈ Adj (x) I L(y) < L(x)}|

D^+(x) = |{ Y∈ Adj (x) I L(y) < L(x)}|

See [7] for proof.

Applications

Permutation graphs can be regarded as a class of intersection graphs in the following manner. Write numbers 1,2,…n horizontally left to right, below that write numbers π_1 π_2……π_n in sequence, again horizontally left to right, draw n straight line segments joining the two 1's, the two 2's etc. We can call this the matching diagram of π. Figure 5 shows such a matching diagram. Notice that the i^{th}segment intersects the j^{th} segment if and only if iand j appear in reversed order in π and this is the same criterion for the vertices i and j of G(π) to be adjacent

Application 1

Suppose we have two collections of cities, the X cities and the Y cities. They are respectively lying on two parallel lines. Also suppose that they are airline routes connecting various X cities with various Y cities, all scheduled to be utilized at the same time of the day our intension is to assign altitude each flight path so that intersecting routes will be at different altitudes and there will not be any midair collisions. We can recognize this as a colouring problem.

This information gives us with a bipartite graph embedded in the plane, as pictured in Figure 6; We number the flight paths by traversing the northern cities from west to east . From this we can extract a matching diagram or corresponding permutations graphG[π] . Assigning altitude to the flight paths so that intersecting paths receive different attitude is equivalent to colouring the vertices of G[π] so that adjacent vertices receive different colours. We can use Algorithm 1 for this colouring.

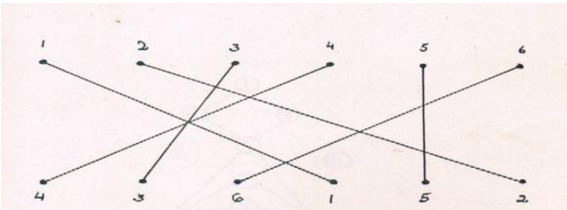


Figure 5

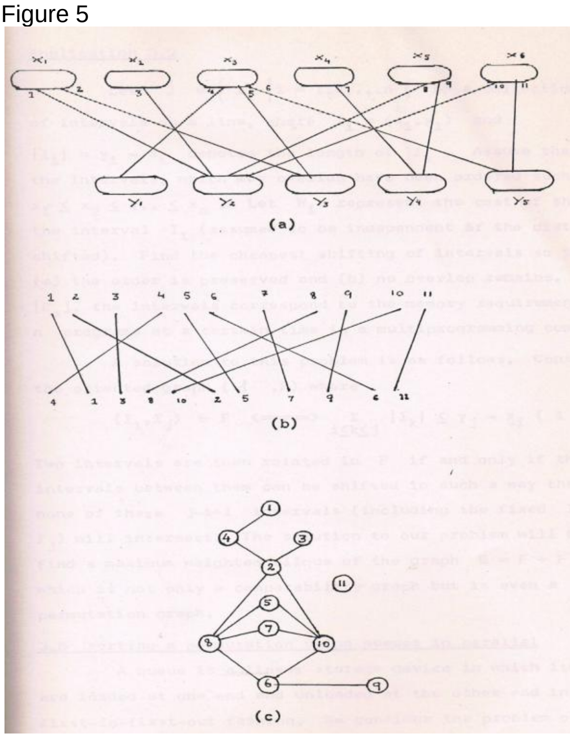


Figure 6

Application 2

Let $\mathcal{I} = \{ I_i / i = 1,2,...n,\}$ be a collection of intervals on a line, where $I_i = (x_i, y_i)$ and $|Ii| = y_i - x_i$ denotes the length of I_i . Assume that the intervals, which may overlap have been ordered such that $x_1 \le x_2 \le \dots\dots\dots\dots \le x_n$ .Let W_i represent the cost of shifting the interval I_i (assumed to be independent of the distance shifted). Find the cheapest shifting intervals so that(a) the order is preserved and (b) no overlap remains. The intervals correspond to the memory requirements of n programs at a certain time in a multiprogramming computer [4] . A solution to this problem is as follows. Consider the oriented graph $(\mathcal{I}, F.)$ where $(I_i, I_j) \in F \Leftrightarrow \sum_{i\le k\le j} |Ik| \le y_j - x_i$, (i<j)Two intervals are then related in F if and only if the intervals between them can be shifted in such a way that none of these j - i + l intervals (including the fixed I_i and I_j) will intersect. The solution to our problem will be to find a maximum weighted clique of the graph $E = F + F^{-1}$which is not only a comparability graph but is even a permutation graph.

5Sorting a permutation using queues in parallel [7]

A queue is a linear storage device in which items are loaded at one end and unloaded at the other end in a first-in-first-out fashion. We consider the problem ofsorting a permutation π of the numbers 1,2………..n using a network of k queues arranged in parallel. Figure 7 depicts a network of k queues in parallel. The permutation sits in the input queue initially, each number passes along to one of the k queues where it is stored temporarily until it is moved onto the output queue. We assume that each queue has unbounded capacity and backing up along an edge by keeping with its direction only. As the number of queues are limited, we will be led to the following problem . Given a network of k queues in parallel, characterize the permutations which can be sorted on it. Given a permutation π, how many queues will we need? Also, find optimal sorting method .

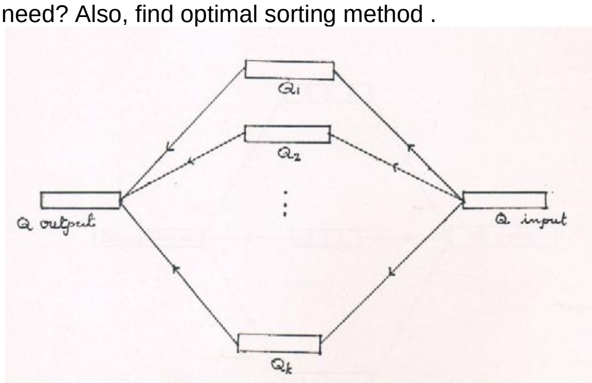


Figure.7

Example: Let π = [4,3,6,1,5,2] .The 4 is placed in Q_1 , the 3 cannot go in Q_1 because it will be forever stuck behind 4 , so put it in Q_2 . The number 6 can either behind 4 on Q_1 or behind 3 onQ_2 .We put 6 behind Q_1 . The 1 must go in Q_3 . The 5 cannot go on Q_1 because 6 is already there. Now put 5 on Q_2 behind 3 . Finally 2 can not go in Q_1 or Q_2 .but it can go to Q_3 . Now all the numbers are stored and we can unload numbers 1-6 from their respective storage places. Figure 8 shows a network which is sorting π = [4,3,6,1,5,2]. From this example we can notice that two numbers to go into different queues when they occur in reverse order in π. Thus ,ifi and j are adjacent in G[π] , then they must go through different queues.

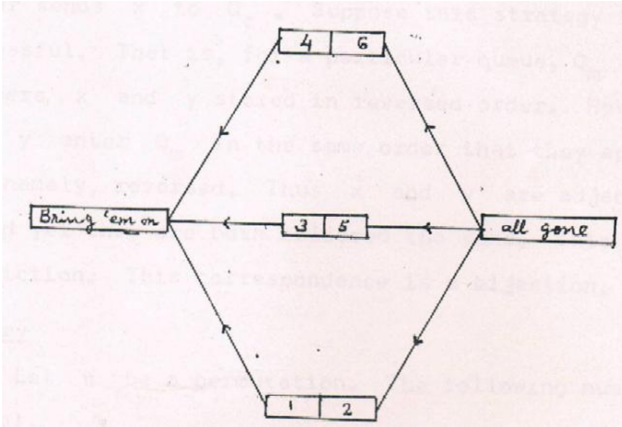


Figure.8

Proposition 1

Let π = [π_1 π_2 ………π_n] be a permutation of the integers {1,2, ... n .} There is a one to one correspondence between the proper k-colouring of G[π] and the successful sorting strategies for π in a network of k parallel queues.

Proof

Assign painters to each Q_1, each with a different colour paint. How sort π in the k-network and everynumber painted as it enters its corresponding queue. The connected vertices i and j receive different colours because i and j of G[π] pass through different queues.

Conversely, given a proper colouring of G[π] using colours 1,2,...K, assign a traffic director to theinput queue. If the colour of x is c, than the traffic director sands x to Q_c. Suppose this strategy is unsuccessful. That is, for a particular queue, Q_m has a pair of numbers x and y stored in reversed order. However, x and y enter Q_m in the same order that they appear in π namely, reversed. Thus x and y are adjacent in G[π] and yet they are both coloured the same, This gives a contradiction. This correspondence is a bijection.

Corollary:

Let π be a permutation. The following numbers are equal.

(i) the chromatic number of G[π].
(ii) the minimum number of queues required to sort π.
(iii) the length of the longest decreasing subsequence or π.

The Canonical Sorting Strategy for π places each number in the first available queue from this strategy, we obtain the Conical Colouring of G[π] and it give a minimal Colouring.

6Algorithm :1 CanonicalColouring of a permutation. [10],[11].

Input : A permutation π = [π_1 π_2………..π_n] of the numbers {1,2,………..n}

Output : A Colouring of the vertices of G[π] of numbers {l,2,…n}

Method : During the J^{th} iteration, π_j is transferred onto the queue Q_i having smallest index i satisfyingπ_j ≥ last entry of Qi We do not actually save the entire contents of Qi An array LAST(i) holds the last number in Q_i . The counter k keeps track of the actual numbar of colours (queues) used. The algorithm is as follows.

begin
1. K← 0
2. for j ←1 to n do
begin
3. i←FIRST allowable queue ;
4. COLOUR (π_j) ← 1 ;
5. LAST (i) ←π_j ;
6. K ←max {K, i} ;
end
7 $\mathcal{X}$ ←K ;
end.

In order to execute statement 3 efficiently, a type of binary insertion can be used. Such a subroutine is given in Figure 9 `

P
rocedure FIRST allowable queue
begin
i←1 ; t ← K + 1 ;
Until i = t do
begin
r← { (1 + t) /2 } ;
If π_j ≥ LAST (r) then t ← r ;
Elsei←- r + 1;
end
returni;
end.

Figure 9

The following theorem shows the correctness of this algorithm.

Theorem 3. [5]

Let π be a permutation of the numbers {1,2………..n}. The canonical colouring of G[π] , as produced by Algorithm 1, is a minimum colouring.

Proof

Algorithm 1 clearly produces a proper$\mathcal{X}$ -coloring of G[π]. We want to show that $\mathcal{X}$=$\mathcal{X}$(G[π]). By corollary of preposition 1, it is sufficient to show that π has a decreasing subsequence of length$\mathcal{X}$.Define a function p as follows . If colour (π_j) = i≥ 2. then $\pi_{p(j)}$ equals the value of LAST (i -1) during the j^{th} iteration. Because $\pi_{p(j)}$ sitting on Q_{i-1} which forced π_j to go down to Q_i . we have $\pi_{p(j)}$> π_j and p_j< j. Then π_{j1}, π_{j2}, ……$\pi_{j\chi}$ is the desired subsequence where COLOUR($\pi_{j\chi}$) = $\mathcal{X}$and π_{j1-1} = $\pi_{p(ji)}$ for$\mathcal{X}$, $\mathcal{X}$-1, $\mathcal{X}-2$, ….. ……2.

Remark

To find a minimum clique cover of G[π] apply algorithm 1 to the reversal π^p of π.The above algorithm can be used to colour any permutation graph G in time proportional to n log n provided we have the permutation π and the isomorphism G →

G[π] . This complexity is independent of the number of edges of G.

Conclusion

This paper studied the permutation graph as a special category of intersection graph and perfect graph .Its characterization by Pnueli Amir's theorem through comparability graph.It also gives permutation labelling by Gill M.K & Acharya B.D'stheorem. It discusses two real life applications and sorting a permutation using queues in parallel. An algorithm for canonical colouring of permutation graph and its complexity were also given. Since the permutation graph is a perfect graph, algorithms can be executed deterministically.

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